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# RESULTS ON PANCYCLC GRAPHS AND COMPLETE HAMILTONIAN GRAPHS

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**Abstract - In 1856, Hamiltonian introduced the Hamiltonian Graph where a Graph which is covered all the vertices without repetition and end with starting vertex. In this paper I would like to prove that Relation between Complete Hamiltonian Graph and Pan-Cyclic Graphs**

**Key Words**: Graph, Complete Graph, Bipartite Graph Hamiltonian Graph and Pan-Cyclic Graphs.

## I. INTRODUCTION

The origin of graph theory started with the problem of Koinsber bridge, in 1735.

This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph.

In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems.

The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits.

In 1852, Thomas Gutherie found the famous four color problem.

Then in 1856, Thomas. P. Kirkman and William R.Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once.

In 1913, H. Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken.

This time is considered as the birth of Graph Theory.

Cayley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory.

Any how the term “Graph” was introduced by Sylvester in 1878 where he drew an analogy between “Quantic invariants” and covariants of algebra and molecular diagrams.

In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory.

In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

In 1971 R.Halin introduced an example of minimally 3- connected Graphs.

**1.1 Definition:** A graph – usually denoted G(V,E) or G = (V,E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

**1.2 Definition:** Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

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**1.3 Definition:** Edges are also known as lines and (in social networks) as ties or links. An edge

e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

**1.4 Example:** The graph depicted in Figure 1 has vertex set V={a,b,c,d,e.f} and edge set

E = {(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)}.

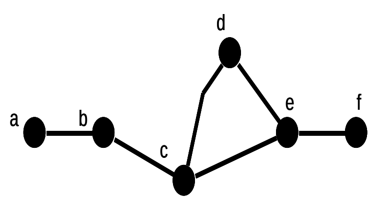


Figure 1.

**1. 5 Definition:** Two vertices *u* and *v* are *adjacent* if there exists an edge (*u,v*) that connects them.

**1.6 Definition:** An edge (u,v) is said to be *incident* upon nodes u and v.

**1.7 Definition:** An edge e = (u,u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

**1.8 Definition:** Every graph has associated with it an *adjacency matrix*, which is a binary *n*□*n* matrix A in which $a_{ij}$ = 1 and $a_{ji}$ = 1 if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

| | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| a | 0 | 1 | 0 | 0 | 0 | 0 |
| b | 1 | 0 | 1 | 0 | 0 | 0 |
| c | 0 | 1 | 0 | 1 | 1 | 0 |
| d | 0 | 0 | 1 | 0 | 1 | 0 |
| e | 0 | 0 | 1 | 1 | 0 | 1 |
| f | 0 | 0 | 0 | 0 | 1 | 0 |

Adjacency matrix for graph in Figure 1

**1.9 Definition:** Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

**1.10 Definition:** While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

**1.11 Note:** Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.

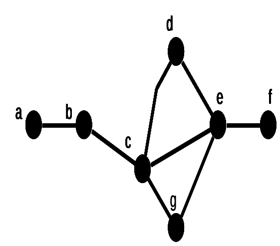


Figure: 2

**1.12 Definition :** A walk is closed if $v_o = v_n$.*degree* of the vertex and is denoted d(v).

**1.13 Definition :** A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

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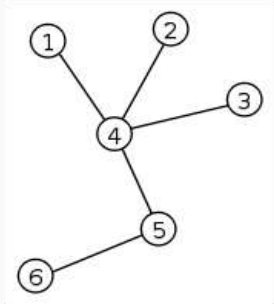


Figure 3: A labeled tree with 6 vertices and 5 edges

**1.14 Definition:** A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

**1.15 Definition:** The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a,b,c,d,e* has length 4.

**1.16 Definition:** The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

**1.17 Definition :** In the mathematical field of graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

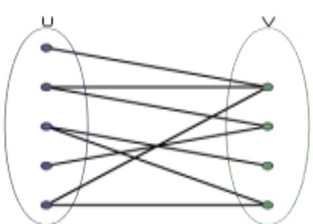


Figure 4: Example of a bipartite graph.

**1.18 Definition:** An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

**1.19 Example :** Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

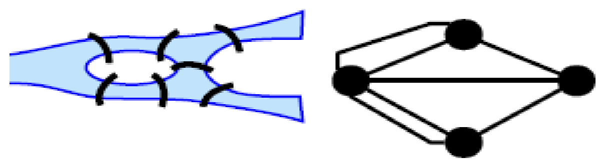

Figure 5**:** Konigsberg problem

## II. HAMILTONIAN GRAPHS, COMPLETE GRAPHS, PAN-CYCLIC GRAPHS

**In this section we have to prove that Relations between Hamiltonian Graphs and Pan-cyclic Graphs.**

**2.1 Definition:** A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be

**2.2 Example:**

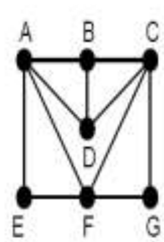


Figure 6: Hamilton Circuit would be AEFGCDBA

**2.3 Definition: Pancyclic Graphs :**A graph G of order n ($\geq$ 3) is pancyclic if G contains all cycles of lengths from 3 to n. G is called vertex-pancyclic if each vertex v of G belongs to a cycle of every length 'l,' where 3 $\leq$ l $\leq$ n.

**2.4 Example** :Clearly, a vertex-pancyclic graph is pancyclic. However, the converse is not true.

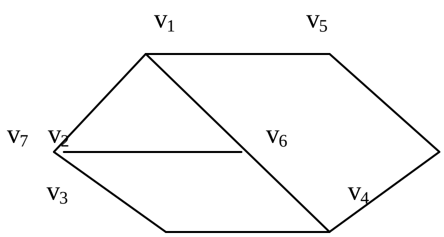


Figure7: Pan-Cyclic Graph

The result of pancyclic graphs was initiated by Bondy , who showed that Ore's sufficient condition for a graph G to be Hamiltonian .

Note that if $\delta \geq n / 2$ then $m \geq n^2 / 2$ .

The proof of the following result due to Thomassen can be found in Bollobas.

**2.5 Theorem :** Let G be a Complete Hamiltonian Graph on 'n' vertices with at least $n^2 / 2$ edges. Then G is either Pancyclic or else is the complete bipartite graph $K_{n/2}, n/2$ .

If G is Complete Hamiltonian and $m > n^2 / 4$ then G is pancyclic.

**Proof** :The result can easily be verified for n = 3.

We may therefore assume that $n \geq 4$.

We apply induction on n.

Suppose the result is true for all graphs of order at most $n - 1$ ($n \geq 4$), and let 'G' be a Complete Hamiltonian Graph of order n.

First, assume that 'G' has a cycle $C = v_0 v_1 \ldots v_{n-2} v_0$ of length $n - 1$.

Let 'v' be the (unique) vertex of 'G' not belonging to 'C'.

If $d(v) \geq n/_2$ , v is adjacent to two consecutive vertices on C and hence 'G' has a cycle of length 3.

Suppose for some r, $2 \leq r \leq (n-1) / _2$ ,

'C' has no pair of vertices 'u' and 'w' on 'C' adjacent to 'v' in G with $d_C(u, w) = r$.

Then if $v_{i_1}, v_{i_2}, \ldots v_{i_{d(v)}}$ are the vertices of 'C' that are adjacent to 'v' in 'G' (recall that C contains all the vertices of G except v),

then $v_{i_1+r}, v_{i_2+r}, \ldots, v_{i_{d(v)}+r}$ are nonadjacent to v in G,

where the suffixes are taken modulo ($n - 1$).

Thus, $2d(v) \leq n - 1$, a contradiction.

Hence, for each r, $2 \leq r \leq (n-1) / _2$ ,

'C' has a pair of vertices 'u' and 'w' on 'C' adjacent to 'v' in G with $d_C(u, w) = r$.

Thus for each r, $2 \leq r \leq (n-1) / _2$ , G has a cycle of length r + 2 as well as a cycle of length

$n - 1 - r + 2 = n - r + 1$ .

Thus 'G' is pancyclic.

If $d(v) \leq (n-1) / 2$, then G[V (C)], the sub graph of 'G' induced by V (C) has at least

$n2/4 - (n-1) / 2 > (n-1)2 / 4$ edges.

So, by the induction assumption, G[V (C)] is pancyclic and hence 'G' is pancyclic.

Next, assume that 'G' has no cycle of length $n - 1$.

Then G is not pancyclic.

In this case, we show that G is $K_{n/2}, n/_2$ .

Let $C = v_0 v_1 v_2 \ldots v_{n-1} v_0$ be a Hamilton cycle of 'G'.

We claim that of the two pairs $v_i v_k$ and $v_{i+1} v_{k+2}$ (where suffixes are taken modulo n), at most only one of them can be an edge of G.

Otherwise, $v_k v_{k-1} v_{k-2} \ldots v_{i+1} v_{k+2} v_{k+3} v_{k+4} \ldots v_i v_k$ is an ($n - 1$)-cycle in G.

It is a contradiction.

Hence, if $d(v_i) = r$, then there are 'r' vertices adjacent to $v_i$ in G and hence at least 'r' vertices (including $v_{i+1}$ since $v_i v_{i-1} \in E(G)$) that are nonadjacent to $v_{i+1}$.

Thus, $d(v_{i+1}) \leq n - r$ and $d(v_i) + d(v_{i+1}) \leq n$.

Summing the last inequality over i from 0 to $n - 1$, we get $4m \leq n^2$.

But by hypothesis, $4m \geq n^2$.

Hence, $m = n^2/4$ and so n must be even.

This gives $d(v_i) + d(v_{i+1}) = n$ for each i, and thus for each i and k, exactly one of $v_i v_k$ and $v_{i+1} v_{k+2}$ is an edge of G.

Thus, if $G \neq K_{n/2}, _{n/2}$ , then certainly there exist i and j such that $v_i v_j \in E$ and $i \equiv j \pmod 2$.

Hence for some j, there exists an even positive integer s such that $v_{j+1} v_{j+1+s} \in E$.

Choose 's' to be the least even positive integer with the above property.

Then $v_j v_{j+1+s}$ does not belongs to E. Hence, $s \geq 4$ (as s = 2 would mean that $v_j v_{j+1} \notin E$).

Again, by , $v_{j-1} v_{j+s-3} = v_{j-1} v_{j-1+(s-2)} \in E(G)$ contradicting the choice of s.

Thus, $G = K_{n/2}, n/_2$ . The last part follows from the

fact that
$|E(K_{n/2}, n/_2, )| = n2/ 4$. Hence the theorem

**2.6Theorem** : Let G ≠ $K_{n/2}$ , $n/_2$ , be a Complete Hamiltonian graph with n □ 3 vertices and let d(u)+d(v) □ n for every pair of non-adjacent vertices of 'G'. Then G is pancyclic.

**Proof :**we know that Let G be a Complete Hamiltonian Graph with n vertices and let u and v be non-adjacent vertices in G such that d(u)+d(v) ≥ n.

Let G+uv denote the super graph of G obtained by joining u and v by an edge. Then G is Hamiltonian if and only if G+uv is Hamiltonian.

We show that G is pan-cyclic by first proving that m □ n2 / 4 and then invoking above theorem This is true if □ □ n / 2 (as 2m = □ $d_i$ □ □ n □ $n^2$/2).

So assume that □ < n / 2

Let S be the set of vertices of degree □ in G.

For every pair (u, v) of vertices of degree □ , d(u) + d(v) < n/2 + $n/_2$ = n.

Hence by hypothesis, 'S' induces a clique of G and |S| □ □ + 1.

If |S| = □ + 1, then G is disconnected with G[S] as a component, which is impossible (as G is Hamiltonian).

Thus, |S| □ □ .

Further, if v □ S, v is nonadjacent to n □ 1 □ □ vertices of G.

If 'u' is such a vertex, d(v) + d(u) □ n implies that d(u) □ n □ □ .

Further, 'v' is adjacent to at least one vertex w ¢ S and d(w) □ □ + 1, by the choice of S.

These facts give that 2m = □ $d_i$ □

(n □ □ □ 1)(n □ □ ) + □ $^2$ + (□ + 1),

where 1≤ i ≤ n the last (□ + 1) comes out of the degree of w.

Thus, 2m □ $n^2$ □ n(2□ + 1) + 2□ $^2$ + 2□ + 1,

which implies that

4m □ $2n^2$ □ 2n(2□ + 1) + 4□ $^2$ + 4□ + 2

= (n □ (2□ + 1)$^2$ + $n^2$ + 1

□ $n^2$ + 1, since n > 2□ .

Consequently, m > $n^2/4$ , and by above theorem,

G is pancyclic.

Hence the theorem

## III. REFERENCES

[1] Bondy J.A. and V. Chvátal, A Method in Graph Theory, Discr. Math. 15 1976), pp 111-136.

[2] Dirac G.A., Some Theorems on Abstract Graphs, Proc. Lond. Math. Soc. 2 (1952), pp 69-81.

[3] Relation between Barnett's Graph and Sparse Graphs, by Venu Madhava Sarma.S and T.V. Pradeep Kumar International Journal of Mathematical Archive-2(12), 2011,Page 2538-2542.

[4] Applications of Graph Theory and its Applications by Venu Madhava Sarma.S International Journal of Computer Application, Issue 2, Volume 1(February 2012) ,
Page 21-31

[5] Some Fundamental Results in Inference and Median Graphs, by Venu Madhava Sarma.S and T.V.Pradeep Kumar International J. of Math. Sci. & Engg. Appls. (IJMSEA) ISSN 0973-9424, Vol. 6 No. III (May, 2012), pp. 47-54

[6] A Note On Planar And Sparse Graphs by Venu Madhava Sarma.S and T.V.Pradeep Kumar proceedings of Two Day UGC National seminar on "Modern Trends in Mathematics and Physical Sciences ( NSMTMPS - 2012) dated 20th , 21st Jan, 2012.

[7] A Note On Planar And Median Graphs by Venu Madhava Sarma. S and T.V Pradeep Kumar Proceedings of the Two days National Conference on Mathematical Sciences and Computational sciences organized by Department of Mathematics Adikavi Nannyya University sponsored by Andhra Pradesh State Council of Higher Education with ISBN:978-81-923777-3-5 page no: 119-120.

[8] West D.B, Introduction to Graph Theory, Prentice-Hall, Inc., New Jersey.

[9] Williams – Nash, C. St. J .A. "Hamiltonian Arcs and Circuits" in recent trends in graphtheory ed. By M.Capobianco et al., Springer – verlag, Berlin, 1971.

[10] West.D.B Introduction to Graph Theory, 2nd. Edition. Prentice Hall, Upper Saddle River, New Jersey, 2001.

[11] Wigderson.A The Complexity Of The Hamiltonian Circuit Problem For Maximal Planar Graphs. Technical Report, Princeton University, Dept. of EECS, Vol. 298, February, 1982.

[12] Tutte. W.T, On Hamiltonian Circuits. J. London Math. Soc., 21, 98-101, 1946.
[13] Tutte .W. T, A Theorem on Planar Graphs. Trans. Amer. Math. Soc., 82, 99-116, 1956.
[14] Tutte. W. T on the 2-Factors of Bicubic Graphs. Discrete Mathematics, 1(2) , 203-208, 1971.
[15] Vandegriend.B Finding Hamiltonian Cycles: Algorithms, Graphs and Performance. Master's thesis, University of Alberta, 1998